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LARGE DISCREPANCY IN HOMOGENEOUS QUASI-ARITHMETIC PROGRESSIONS

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We prove that the class of *homogeneous quasi-arithmetic progressions* has unbounded discrepancy. That is, we show that given any 2-coloring of the natural numbers and any positive integer D, one can find a real number $\alpha \geq 1$ and a set of natural numbers of the form $\{0,[\alpha],[2\alpha],[3\alpha],\dots,[k\alpha]\}$ so that one color appears at least D times more than the other color. This was already proved by Beck in 1983, but the proof given here is somewhat simpler and gives a better bound on the discrepancy.

1. Introduction

A special case of Van der Waerden's Theorem [4] is that given any 2-coloring of the natural numbers, it is possible to find monochromatic arithmetic progressions of arbitrary finite length. Around the year 1933, Paul Erdős asked what would happen if one 2-colored the natural numbers, and restricted one's attention to those arithmetic progressions which begin at zero, called *homogeneous* arithmetic progressions. It is easy to give a 2-coloring of the natural numbers so that no homogeneous arithmetic progression is monochromatic: Color $n \in \mathbb{N}$ "red" if $n=2^k m$ where k and m are odd, otherwise color n blue. Already the second and third term in every arithmetic progression have different colors. So Erdős asked if it was possible to color $\mathbb{N}$ so well that each homogeneous arithmetic progression will contain roughly the same number of each color, say, within 100. Or is it true that under every 2-coloring of $\mathbb{N}$, one can find arbitrarily unbalanced homogeneous arithmetic progressions?

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More precisely, given a function $f : \mathbb{N} \to \{-1,+1\}$, define the *discrepancy of* f to be $d(f) = \sup_{k\geq 1} \sup_{t\geq 0} \left|\sum_{i=0}^{t} f(ik)\right|$. Erdős is asking for a function with finite discrepancy. This problem is still open. In their survey article [2], Erdős and Graham note the existence of a function f satisfying

$$\sup_{k\geq 1} \sup_{0\leq t\leq n/k} \left|\sum_{i=0}^{t} f(ik)\right| \leq c \cdot \log n.$$

Motivated by some discussion on quasi-arithmetic progressions in the above-mentioned survey article (there called *generalized* arithmetic progressions), J. Beck showed [1] that with respect to this larger class of sequences, no 2-coloring of $\mathbb{N}$ can have bounded discrepancy. In that paper, Beck adapted a technique that W. M. Schmidt had used to improve upon a result of Roth concerning uniform distributions of points in the unit square. In this paper, we use Roth's original technique [3] to obtain a more elementary proof of an (in some sense) improved result.

2. Homogeneous Quasi-Arithmetic Progressions and the Main Theorem

Define a *quasi-arithmetic progression* A to be a set of natural numbers of the form $\{0, [\alpha], [2\alpha], \dots, [m\alpha]\}$ where $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. Here $[\cdot]$ denotes the greatest integer function and α is called the *common difference*. Note that it is easy to construct a homogeneous quasi-arithmetic progression with large discrepancy; simply let α be very small, so that the sequence begins with as many terms as you wish all having the same color as 0. Thus we restrict our attention to the case $\alpha \geq 1$. In this case, we define the discrepancy D of f with respect to quasi-arithmetic progressions as:

$$D(f) = \sup_{\alpha\geq 1} \sup_{t\geq 0} \left|\sum_{i=0}^{t} f([i\alpha])\right|.$$

The main result of this paper is the following:

Theorem 2.1 (Main Theorem). *Given any function $f : \mathbb{N} \to \{-1,+1\}$, and any integer $t \geq 1$, then for all sufficiently large n, there is some homogeneous quasi-arithmetic progression A with common difference between t and $t+1$, and largest term less than n, such that*

$$\left|\sum_{a\in A} f(a)\right| > \frac{1}{24\sqrt{t}} \cdot (\log_2 n)^{1/4}.$$

Beck [1] had already proved the following:

Theorem 2.2 (Beck). *Given any function* $f:\mathbb{N}\to\{-1,+1\}$, *then for almost every* $\alpha\geq 1$,

$$\sup_{m\in\mathbb{N}}\left|\sum_{k=1}^{m} f([k\alpha])\right|=\infty.$$

Among all homogeneous quasi-arithmetic progressions with largest term at most n, Beck's proof yields one having discrepancy at least $\log^* n$. Our Main Theorem is an improvement in that it finds larger discrepancy, but we do not get the result that "almost all" homogeneous quasi-arithmetic progressions achieve arbitrary discrepancy when extended indefinitely.

The values ± 1 of f will be referred to as 'colors' throughout. Here we quickly outline the proof of Theorem 2.1. A homogeneous quasi-arithmetic progression can be described by the ordered pair of parameters: (*number of terms, largest term*), thought of as a point in the plane. Given the function f, we will construct an associated function D defined at every point in the positive quadrant of the (x,y)-plane, and which gives at each point (x,y) with integer x, the sum of the function values of the terms of the homogeneous quasi-arithmetic progression associated with that point. Next we use an idea of Roth; we construct a family of mutually orthogonal functions in the plane and employ Bessel's inequality to show that D is large in the $\mathcal{L}_2$-norm, and hence is large at some point that has integer x-coordinate.

3. The discrepancy function $D(x,y)$

Let $f:\mathbb{N}\to\{-1,+1\}$ be given. For all $(a,b)\in\mathbb{N}^2$, we color the unit-length, half-open, vertical line segment $l_{a,b}=\{(x,y): x=a,\ b\leq y<b+1\}$ with the color of $f(b)$. In this fashion, the homogeneous quasi-arithmetic progression $\{0,[\alpha],[2\alpha],\ldots,[k\alpha]\}$ will correspond to the line $y=\alpha x$ in the domain $0\leq x\leq k$ and the expression $\sum_{i=0}^{k} f([i\alpha])$ is simply the sum of the values of those vertical line segments which the line $y=\alpha x$ crossed.

Now let us define $D(s,t)$, $s,t\in\mathbb{R}^+$, to be the sum of the values of the vertical line segments crossed by the line $y=(t/s)x$ in the domain $0\leq x\leq s$. Note that D is constant on any half-open line segment of the form $\{(x,y): y=\alpha x,\ m\leq x<m+1,\ \alpha\in\mathbb{R}^+,\ m\in\mathbb{N}\}$. Thus if we show that $D(x,y)$ is large at some real x, then we have shown that $D(x^*,y^*)$ is large for some integer value x^*. Consider $l_{a,b}$ and suppose it has color $+1$. Notice that $l_{a,b}$ will contribute its $+1$ value to the discrepancy function $D(s,t)$ only at those

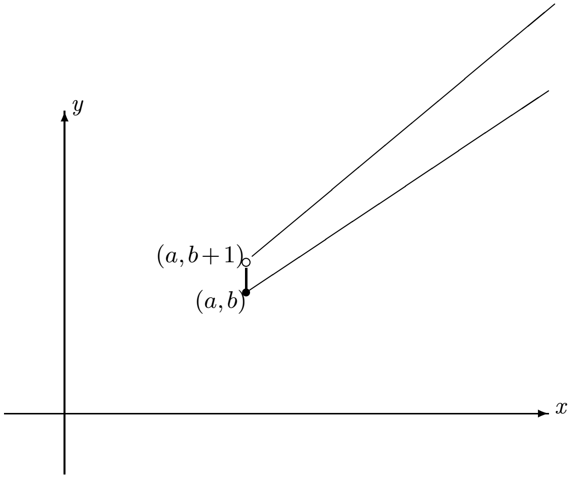

Figure 1. The set of points $B_{a,b}$ behind the line $l_{a,b}$

points (s,t) for which the line segment between the origin and (s,t) passes through $l_{a,b}$ (see Figure 1). That is, the set of points "**B**ehind" $l_{a,b}$:

$$B_{a,b} = \left\{(x,y) : x \geq a,\ \frac{b}{a} \leq \frac{y}{x} < \frac{b+1}{a}\right\}.$$

We define the function $D_{a,b}(x,y)$ to be the function which is the same color as $l_{a,b}$ in that region, and 0 elsewhere. Thus:

$$D_{a,b}(x,y) = \begin{cases} f(b) & \text{if } x \geq a,\ b/a \leq y/x < (b+1)/a \\ 0 & \text{otherwise}, \end{cases}$$

and

$$D(x,y) = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} D_{a,b}(x,y).$$

There is no problem with convergence here, because for each point (x,y), $D_{a,b}(x,y)$ is non-zero for only finitely many pairs (a,b).

4. The test functions $g_i(x,y)$

Given two piecewise-constant functions $f,g : R \to \mathbb{R}$, where R is some region in the plane, we define the *inner product* $\langle f,g\rangle$ by

$$\langle f,g\rangle = \iint_R f(x,y) \cdot g(x,y)\ dx\,dy,$$

and the norm $\|f\|$ of f by

$$\|f\| = \sqrt{\langle f, f\rangle}.$$

A family $\{g_1, g_2, \ldots, g_m\}$ of functions is called *orthonormal* if each g_i has norm 1, and for each $1 \le i < j \le m$, $\langle g_i, g_j\rangle = 0$.

Theorem 4.1 (Bessel's Inequality). *If $\{g_1, g_2, \ldots, g_m\}$ is an orthonormal family of functions, then for any piecewise-constant function D,*

$$\|D\|^2 \ge \sum_{i=1}^{m} \langle D, g_i\rangle^2.$$

We would like to construct the g_i so that they form an orthonormal family, with the inner products $\langle D, g_i\rangle$ large. We will construct a function g_i for each $i \in \{1, 2, \ldots, m\}$, where m will be determined later, but will be on the order of $\log_2 n$. In what follows, let i be fixed, and let us construct the function g_i.

Assume that n is a power of 2. Let $l = 2^i$ and $h = 1/(\beta \cdot 2^i)$ where β will be determined later, and is assumed to be greater than 1. For a given natural number t, let $R = R(n, t)$ be the region in the first quadrant bounded by the four lines $y = tx$, $y = (t+1)x$, $x = (n/2)$, $x = n$. (Note that points in R correspond to homogeneous quasi-arithmetic progressions with largest term at most $n(t+1)$ instead of n. We will adjust for this at the end.) We subdivide R into trapezoids in the following way: Let a family of vertical lines spaced distance l apart divide R into vertical strips, and then let lines through the origin whose slopes are spaced h/n apart divide these strips into trapezoids. (See Figure 2.) (We call this grid of trapezoids the *i'th trapezoid grid*.) The latter family intersects the right boundary of R in a set of points evenly spaced h apart. Thus l is the length of each trapezoid (in the x-direction) and h represents the approximate vertical thicknesses of the rightmost trapezoids. We call h/n the *slope width* of the trapezoids. Notice that we only gave the spacing of these two families without telling exactly how this grid of trapezoids would be positioned. This is intentional. The exact position will be left undecided until the very end of the proof where we will exploit randomness in the placement of this grid to show that the $\langle D, g_i\rangle$ can have a large square-sum.

We now construct the function g_i trapezoid by trapezoid. By a *switch point* in R, we will mean a point of R with integer coordinates (a, b) such that $f(b-1) \ne f(b)$. We call (a, b) a *positive* switch point if $f(b) = +1$, and we call (a, b) a *negative* switch point if $f(b) = -1$.

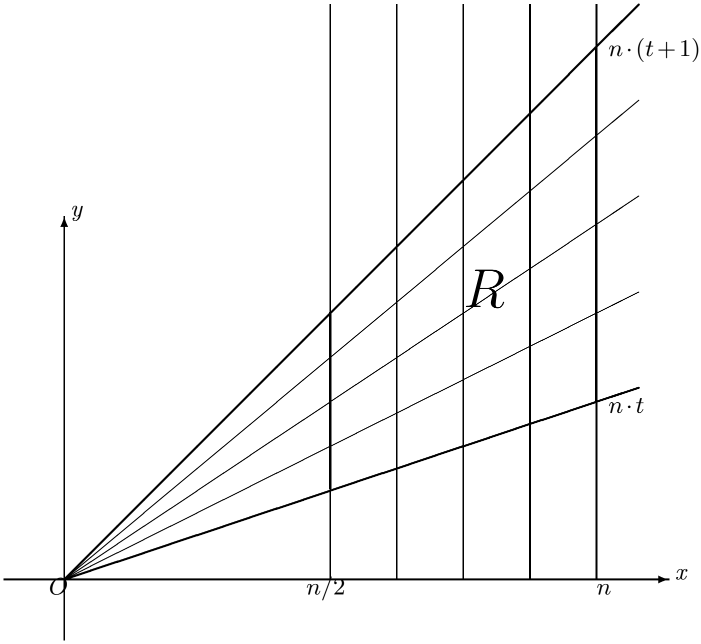

Figure 2. The grid in the setup of g_i. (Note: not drawn to scale.)

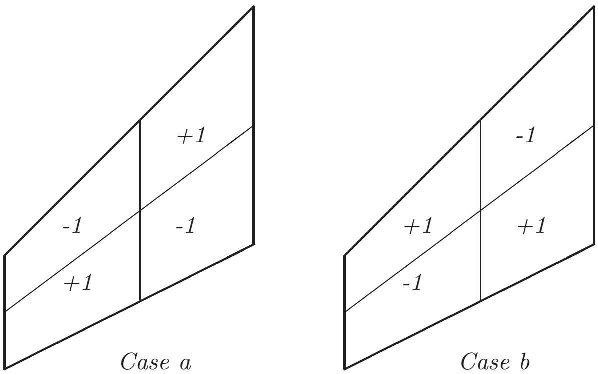

Figure 3. Two choices for G_i inside each trapezoid. Note that the vertical line bisects the *area* of the trapezoid, not its sides

We first construct functions G_i which are not normal, and then let $g_i = \frac{G_i}{\|G_i\|}$. We define the function G_i as follows: If a trapezoid T contains no switch point, or if it contains more than 1 switch point, then G_i is identically 0 inside that trapezoid. Also, G_i is identically 0 inside trapezoids which intersect the boundary of R. If a trapezoid T contains exactly one switch point (a, b), we divide T with two line segments: A vertical line segment which bisects the area of T, and a sloped line segment that bisects the left and right sides of T, and hence also bisects the areas of the two small trapezoids formed by the first cut. Thus T is divided into 4 regions of equal area. We choose from one of 2 ways to define G_i inside that trapezoid (see Figure 3). If (a,b) is a positive switch point, then we choose G_i in T so that the upper right quadrant is positive. If (a,b) is a negative switch point, then we choose G_i in T so that the upper right quadrant is negative. Note that this implies that $\int\int_T G_i \cdot D$ is never negative. If we let A_i denote the sum of the areas of those trapezoids for which G_i is non-zero, then

$$\|G_i\| = \sqrt{A_i}.$$

5. The sum of the $\langle g_i, D\rangle^2$ can be made large

We choose the horizontal displacement of the vertical lines in the i'th trapezoid grid randomly, so that the x-coordinate of the rightmost vertical line which cuts R is distributed uniformly in the interval $[n-l,n)$, and choose the slope displacement of the pencil of lines through the origin randomly, so that the slope of the lowest of them which cuts R is distributed uniformly in the interval $[t,t+h/n)$. Thus the position that a switch point occupies within its trapezoid is random; uniformly distributed in the x direction, and, given its x-coordinate, uniformly distributed vertically within the trapezoid. We call a switch point p a *good switch point* (for G_i) if for every placement of the i'th trapezoid grid, p finds itself alone in a trapezoid, and that trapezoid does not intersect any boundary of R. Let sw_i denote the number of good switch points for G_i.

For a good switch point p we can calculate the expected value of $\int\int_T G_i \cdot D$ under a random placement of the grid, where T is the trapezoid containing p. First rewrite

$$\int\int_T G_i \cdot D$$

as

$$\int\int_T G_i \cdot \left(\sum_{a=0}^{\infty} \sum_{b=0}^{\infty} D_{a,b} \right). \tag{5.1}$$

Claim 5.1. *If neither (a,b) nor $(a,b+1)$ lie inside T, then $\int\int_T G_i \cdot D_{a,b} = 0$.*

Proof. It is easy to see that if T lies entirely inside or entirely outside the set $B_{a,b}$, then $\int\int_T G_i \cdot D_{a,b} = 0$. So suppose that T lies partially inside $B_{a,b}$, so that

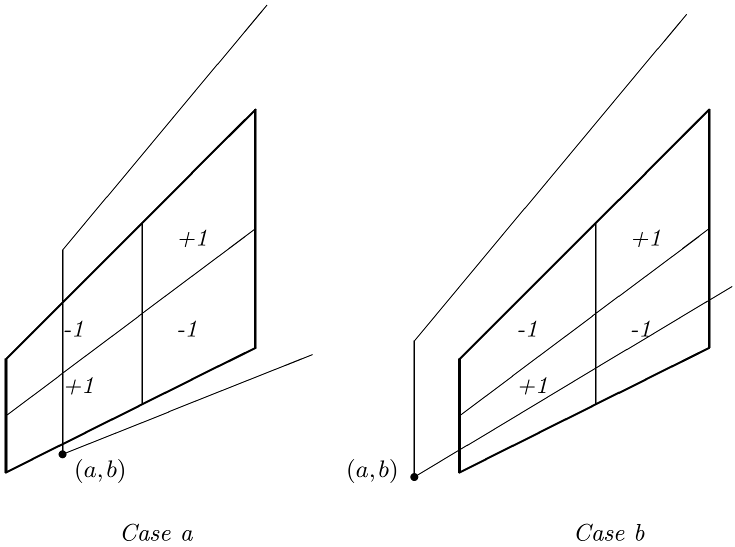

Figure 4. Two ways T can partially intersect $B_{a,b}$ as in the statement of Claim 5.1

T is cut by exactly one of the 3 bounding lines of $B_{a,b}$. The 2 essentially different ways this can happen are shown in Figure 4. In Figure 4a, the line segment $l_{a,b}$ cuts T vertically so that T is divided into 6 regions (sub-trapezoids). The value of $D_{a,b}$ to the left of $l_{a,b}$ is zero. The remaining (2 or 4) sub-trapezoids occur in top-bottom pairs having equal areas, but with G_i having opposite signs in them, so that $\int\int_T G_i \cdot D_{a,b} = 0$. Case 4b is essentially the same, except that the *left-right* pairs will now have equal areas. ∎

Thus the only terms of (5.1) which are non-zero are the (a,b)-term and the $(a,b-1)$-term, where (a,b) is the switch point in T. Let $p=(a,b)$, and define

$$D_p = D_{a,b-1} + D_{a,b}.$$

Then

$$\int\int_T G_i \cdot D = \int\int_T G_i \cdot D_p\,,$$

Assume without loss of generality that p is a positive switch point, so the line segment $l_{a,b}$ above p is colored $+1$, and $l_{a,b-1}$ below p is colored -1, and G_i is as in Figure 5. When p is at the center of t, the value of $\int\int_T G_i \cdot D_p$ is just one half the area of T. If T were a parallelogram, then the value of $\int\int_T G_i \cdot D_p$ would vary linearly as we moved p parallel to one of its sides, reaching 0 at each boundary, so that the average value obtained by $\int\int_T G_i \cdot D_p$, as p varies with T, is $(1/8) \cdot area(T)$. The length l of our trapezoids will never exceed $\sqrt{n}$, so the error we make in assuming T is a

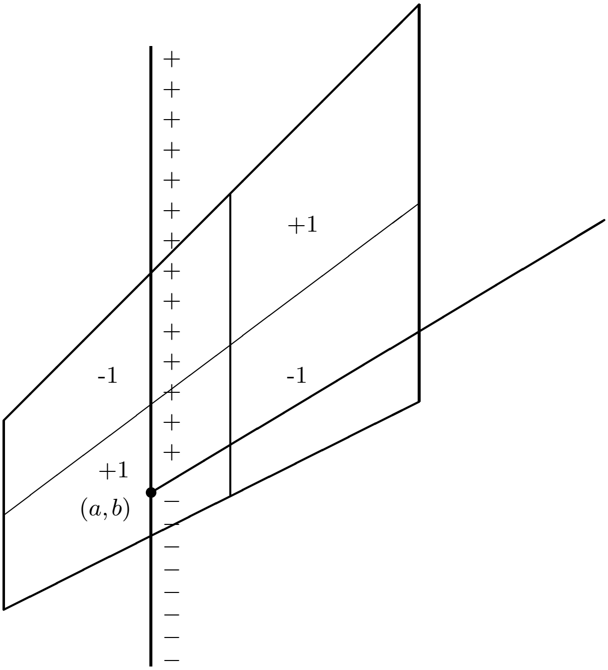

Figure 5. The positive switch point p inside T

parallelogram is minute with respect to the area of T, as some tedious but elementary geometry shows. Replacing $1/8$ with a smaller value, say $1/10$, and letting n be large enough, we obtain:

Claim 5.2. *Given a good switch point p, the expected value of $\int\int_T G_i \cdot D$, where T is the trapezoid containing p, is at least $(1/10) \cdot$ (area of T).* ∎

In the construction of each G_i we use a grid of trapezoids with length $l = 2^i$ and slope width h/n, so each trapezoid has area at least $(n/2) \cdot (h/n) \cdot l = 1/(2\beta)$. That is:

$$\text{Area of any grid trapezoid} \geq \frac{1}{2\beta}.$$

Since each good switch point lies in a trapezoid, we have

$$A_i \geq \frac{1}{2\beta} \cdot sw_i \tag{5.2}$$

The expected value of $\int\int_R G_i \cdot D$ can be written as

$$\mathbf{E}\left[\int\int_R G_i \cdot D\right] = \mathbf{E}\left[\sum_{T \subset R} \int\int_T G_i \cdot D\right]$$

which, using linearity of expectation and Claim 5.2, is equal to

$$\sum_{T\subset R} \mathbf{E}\left[\int\!\!\int_T G_i \cdot D\right] \geq \frac{1}{10}\cdot A_i\,.$$

Finally, since for any random variable X,

$$\mathbf{E}[X^2] \geq \mathbf{E}^2[X]\,,$$

we get that

$$\mathbf{E}\left[\left(\int\!\!\int_R G_i \cdot D\right)^2\right] \geq \frac{1}{100} A_i^2\,. \tag{5.3}$$

Recall that

$$g_i = \frac{G_i}{\|G_i\|} = \frac{G_i}{\sqrt{A_i}}. \tag{5.4}$$

Substituting (5.4) into (5.3) yields:

$$\mathbf{E}\left[\left(\int\!\!\int_R g_i \cdot D\right)^2\right] \geq \frac{1}{100} A_i \tag{5.5}$$

and, by linearity of expectation,

$$\mathbf{E}\left[\sum_{i=1}^{m} \langle g_i, D\rangle^2\right] = \mathbf{E}\left[\sum_{i=1}^{m}\left(\int\!\!\int_R g_i \cdot D\right)^2\right] \geq \frac{1}{100}\sum_{i=1}^{m} A_i\,. \tag{5.6}$$

Consequently, it is enough to show that for every i there are many good switch points. We do that in section 7. In section 6, we show that the g_i are orthogonal. In order to do this, we will require that for each $i = 2, 3, \ldots, m$, the set of vertical lines in the grid for g_i is a subset of the set of vertical lines in the grid for g_{i-1}, and the set of sloping lines for g_{i-1} is a subset of the set of sloping lines for g_i. Thus these families of vertical and sloping lines are nice nested dyadic families which all move together when we randomly position the grids. The loss of independence between the G_i does not affect our result since we used nothing stronger than the linearity of expectation.

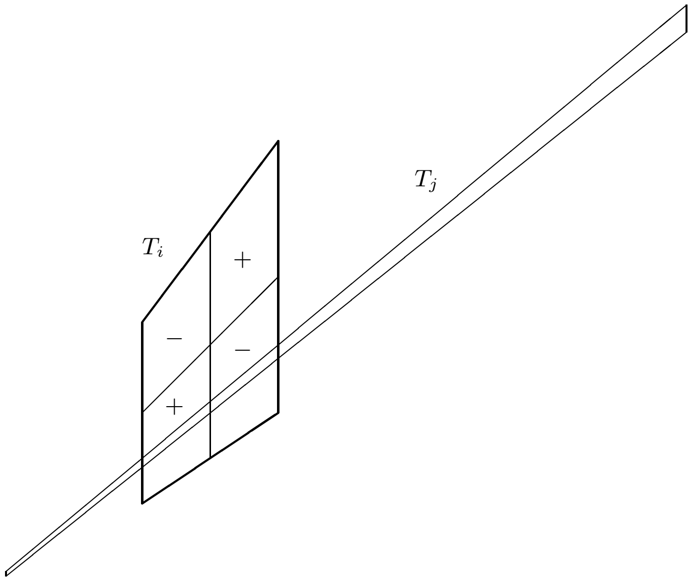

Figure 6. The intersection of T_i and T_j. For clarity, the divisions of T_j into $\pm$ regions are left out.

6. The g_i are orthogonal

The proof that the g_i are orthogonal is similar to the proof of Claim 5.1.

Claim 6.1. *For each* $1 \le i < j \le m$,

$$\int\!\!\int_R g_i \cdot g_j = 0.$$

Proof. The integral can be rewritten as

$$\sum_{T_i} \sum_{T_j} \frac{1}{\sqrt{A_i}\sqrt{A_j}} \int\!\!\int_{T_i \cap T_j} G_i \cdot G_j,$$

where T_i and T_j are trapezoids in the grids used in the construction of g_i and g_j respectively. We distinguish two cases:

Case a. $T_i \cap T_j = \emptyset$. In this case the integral is clearly zero.

Case b. In case they have non-empty intersection, since $i < j$, the trapezoid T_j is the longer of the two trapezoids, while T_i is the thicker (see Figure 6). We chose nested dyadic intervals for the heights of the trapezoids, so for any vertical line L we have $L \cap T_i \cap T_j = L \cap T_j$. If we integrate $\int\!\int_{T_i \cap T_j} G_i \cdot G_j$ in the y-direction first, we clearly get 0 because of the opposite signs within T_j

above and below the sloped line bisecting its area. This is true even if the vertical line bisecting the area of T_j lies within $T_i \cap T_j$. ∎

7. There are many good switch points

The proof that there are many good switch points will proceed as follows: First we find a trivial lower bound on the number of sign changes for the function f, which will give us a lower bound on the number of switch points inside R. Then we use Pick's Theorem to show that for the right choice of β, the total number of lattice points of R which aren't alone inside a trapezoid is less than half the number of switch points inside R.

Suppose we are given a 2-coloring of $\{0,1,\ldots,n\}$. Let M denote the maximum discrepancy of all homogeneous quasi-arithmetic progressions in that set. We assume that

$$M < log^{1/4} n \tag{7.1}$$

otherwise our proof is done. Then, in particular, the homogeneous arithmetic progression $\{0,t,2t,\ldots,n\cdot t\}$ has discrepancy at most M, implying that there cannot be a run of more than $2M$ consecutive $+1$s or -1s in the sequence $f(0), f(t), f(2t), \ldots, f(n\cdot t)$. Therefore:

Lemma 7.1. *If the discrepancy of every homogeneous quasi-arithmetic progression of f having common difference $\alpha \in (t,t+1)$ has discrepancy $\leq M$, then every $2Mt+1$ consecutive values of f must contain at least one sign change.* ∎

This trivial lower bound will suffice to prove Theorem 2.1. Note, however, that if one could replace $2Mt$ by $O(\sqrt{M}t)$, then the exponent $1/4$ of $\log n$ in Theorem 2.1 could be improved to $1/3$. The following corollary follows from the fact that for sufficiently large n, all the lengths of the region R are very large compared to M, the fraction of trapezoids intersecting the boundary of R tends to 0, and from the generous replacement of '$2Mt+1$' from Lemma 7.1 by '$3Mt$'.

Corollary 7.2. *For each g_i, the number of switch points in R which are not in trapezoids intersecting the boundary of R is at least $1/(3Mt)$ times the area of R, i.e., is at least $n^2/8Mt$.* ∎

We fix i, and recall that the grid for the construction of g_i used vertical lines spaced $l = 2^i$ apart, and sloped lines through the origin whose slopes were spaced h/n apart, where $h = 1/(\beta \cdot 2^i)$. Our goal is to show that the

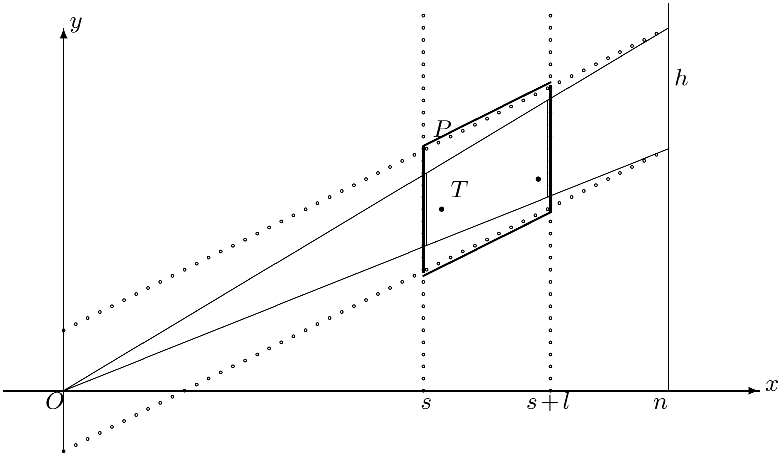

Figure 7. The parallelogram P around T

number of lattice points in the interior of R which, for some placement of this grid, do *not* wind up alone inside a trapezoid, is smaller than half the number of switch points in R.

Consider the typical trapezoid T bounded by the four lines

$$y = \left(\mu - \frac{h}{2n}\right)x, \quad y = \left(\mu + \frac{h}{2n}\right)x, \quad x = s, \quad x = s + l \tag{7.2}$$

as shown in Figure 7. We call μ the *slope of* T. Suppose T contains more than one lattice point. Then so would the parallelogram P which is bounded by the lines

$$y = \mu x - \frac{h}{2}, \quad y = \mu x + \frac{h}{2}, \quad x = s, \quad x = s + l.$$

Let (a, b) be a lattice point in the trapezoid, and translate (a, b) to the origin, together with the trapezoid and parallelogram. Let T' and P' be these translates of the trapezoid and parallelogram, respectively (see Figure 8).

Let $Q(\mu)$ be the parallelogram bounded by the lines

$$y = \mu x - h, \quad y = \mu x + h, \quad x = -l, \quad x = l. \tag{7.3}$$

(We call μ the *slope of* Q.) Note that $Q(\mu)$ is constructed to be big enough to contain P', no matter how P landed after the translation. Thus, in order to show that a point $(a, b) \in R$ will be alone inside its trapezoid for every placement of the grid, it is enough to show that for every μ that could be the slope of a trapezoid containing (a, b), the large parallelogram $Q(\mu)$ contains

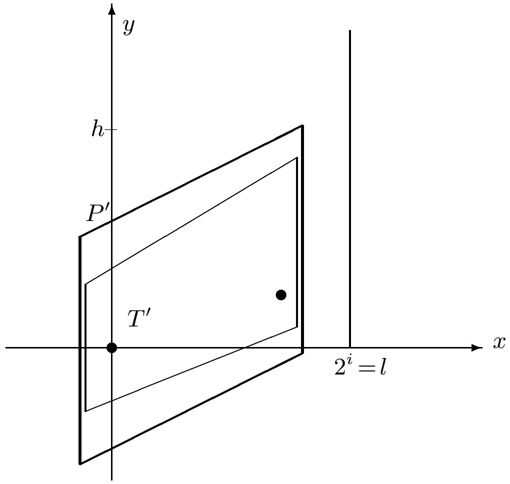

Figure 8. The translated T' and P'

no lattice point other than the origin. Since Q is symmetric with respect to the origin, we need only consider the parallelogram Q' which is the right half of Q, and note that since μ lies in the interval $(t, t+1)$, and Q' is bounded on the right by the line $x = l$, any lattice point (c, d) which lies in Q' must satisfy

$$\begin{gathered} 1 \leq c \leq l \\ c \cdot t + 1 \leq d \leq c \cdot (t+1). \end{gathered} \tag{7.4}$$

Recall that $h < 1$ and note that we have excluded from consideration those lattice points on the line $y = tx$, since they do not lie in the interior of R.

Let us consider the problem from the other direction: Find all μ for which $Q(\mu)$ contains a lattice point other than the origin, and remove from consideration those lattice points $(a, b) \in R$ which are contained in some grid trapezoid of slope μ.

Consider a lattice point (c, d) in $Q'(\mu)$, and recall (7.3). We find

$$Q'(\mu) \ni (c, d) \iff \mu \in \left(\frac{d-h}{c}, \frac{d+h}{c} \right). \tag{7.5}$$

Call slopes, μ, for which $Q'(\mu)$ contains a lattice point other than the origin *bad slopes*, and observe that trapezoids with slopes which are not bad *never* contain more than one lattice point. Now we find an upper bound on the number of lattice points which can lie inside trapezoids with bad slope. For a given bad slope μ, the lattice points which lie in the angular region between the lines

$$y = \left(\mu - \frac{h}{2n}\right)x, \quad y = \left(\mu + \frac{h}{2n}\right)x, \tag{7.6}$$

(from (7.2)) are the ones that we wish to remove from consideration. For a given (c, d), we have the bad slopes from equation (7.5). The union of the angular regions of (7.6) for each of these slopes is another angular region between the lines

$$y = \left(\frac{d-h}{c} - \frac{h}{2n}\right)x, \quad y = \left(\frac{d+h}{c} + \frac{h}{2n}\right)x. \tag{7.7}$$

Since $2n > c$, the angular region of (7.7) is contained in the angular region between the lines

$$y = \left(\frac{d-2h}{c}\right)x, \quad y = \left(\frac{d+2h}{c}\right)x. \tag{7.8}$$

So any lattice point which lies in a trapezoid with a bad slope lies in one of the angular regions of (7.8) for some point (c, d), which covers an interval of slopes of length

$$\frac{d+2h}{c} - \frac{d-2h}{c} = \frac{4h}{c}.$$

Let Δ denote the total measure of the interval $(t, t+1)$ which is covered by these intervals of bad slopes. Recalling the relations (7.4), we calculate

$$\Delta \leq \sum_{c=1}^{2^i} \sum_{d=tc+1}^{(t+1)c} \frac{4h}{c} = \sum_{c=1}^{2^i} 4h = 4h \cdot 2^i = \frac{4}{\beta}. \tag{7.9}$$

Let R^* denote the triangular region bounded by the lines

$$y = tx, \quad y = (t+1)x, \quad x = 2n.$$

The following lemma is implied by the inequality (7.9):

Lemma 7.3. *The area of R^* which is covered by the angular regions of (7.8) is less than $(4/\beta)\cdot$(the area of R^*).* ∎

The fact that this area is a small percentage of the total area of R is not terribly conclusive, for we could cover *every* lattice point of R with angular regions whose total area is arbitrarily small. What's relevant is that the lines (7.8) bounding the bad angular regions have rational slopes with fairly small denominator. In (7.8), d and c are integers, and the denominator $\beta \cdot 2^i$ of h is an integer, so the slopes of these lines have denominator at most $c \cdot \beta \cdot 2^i \leq \beta \cdot 4^i$.

We wish to guarantee that each of the lines in (7.8) contains a lattice point (σ, τ) with $n \le \sigma \le 2n$. This will happen if the denominator of the slopes is less than n, so we set

$$\beta \cdot 4^i \le n.$$

And this must hold true for all $i = 1, 2, \ldots, m$, which will be assured as long as

$$\beta \cdot 4^m \le n. \tag{7.10}$$

Assume that (7.10) holds. For each (c, d) consider the points (σ, τ) and (σ', τ'), $n \le \sigma$, $\sigma' \le 2n$, which lie *on* the lines of (7.8) bounding the angular region. Any lattice point of R lying in this angular region also lies within the triangle with vertices $(0,0), (\sigma, \tau)$ and (σ', τ'). The vertices of this triangle are not in R, and the angular region of (7.8) properly contains the region bounded by the lines (7.7) which contains all the lattice points of R that we wish to remove from consideration. By Pick's Theorem, the area of each of the lattice triangles is an upper bound on the number of lattice points in its interior. But by Lemma 7.3, the sum of these areas is at most $(4/\beta) \cdot 2n^2$. We have proved the following lemma:

Lemma 7.4. *For each i, there are no more than $(8/\beta) \cdot n^2$ lattice points of R which, under any placement of the grid, can share a trapezoid with another lattice point.* ∎

We want this number to be less than half the total number of switch points of R. Recall Corollary 7.2, and set

$$\frac{8n^2}{\beta} \le \frac{1}{2} \cdot \frac{n^2}{8Mt},$$

which is true if we let

$$\beta = 128Mt. \tag{7.11}$$

We can now choose m to satisfy the inequality (7.10). Set

$$\beta \cdot 4^m \le n,$$

which is true if we fix any $\epsilon > 0$ and let

$$m = (1/2 - \epsilon) \log_2 n. \tag{7.12}$$

(We are using (7.1) and the fact that t is fixed.)

8. Conclusion

At least half the switch points from Corollary 7.2 are good, so that

$$sw_i \geq \frac{n^2}{16Mt}.$$

Thus (5.2) and (7.11) give

$$A_i \geq \frac{n^2}{4096M^2t^2},$$

so that (5.6) becomes

$$\mathbf{E}\left[\sum_{i=1}^{m} \langle g_i, D\rangle^2\right] \geq \frac{1}{100}\sum_{i=1}^{m} \frac{n^2}{4096M^2t^2} = \frac{n^2 m}{409600M^2t^2}. \tag{8.1}$$

This means that for some placement of the grids, the g_i that result satisfy

$$\sum_{i=1}^{m} \langle g_i, D\rangle^2 \geq \frac{n^2 m}{409600M^2t^2}.$$

By Bessel's Inequality (Theorem 4.1), we have

$$\|D\|^2 \geq \frac{n^2 m}{409600M^2t^2}.$$

So there is some point $(x,y)\in R$ such that

$$D^2(x,y) \geq \frac{1}{\text{area of } R} \cdot \frac{n^2 m}{409600M^2t^2},$$

that is,

$$D^2(x,y) \geq \frac{8}{3n^2} \cdot \frac{n^2 m}{409600M^2t^2} = \frac{m}{153600M^2t^2},$$

or

$$D(x,y) \geq \frac{\sqrt{m}}{392Mt}.$$

Recalling that M was a tight upper bound for $|D(x,y)|$ in R, and substituting (7.12), we get

$$|D(x,y)| \geq \frac{\sqrt{(1/2-\epsilon)\log_2 n}}{392tD(x,y)},$$

or

$$D^2(x,y) \geq \frac{\sqrt{(1/2-\epsilon)\log_2 n}}{392t},$$

which gives

$$|D(x,y)| \geq \frac{(\log_2 n)^{1/4}}{23.6\sqrt{t}}.$$

The ϵ is absorbed by the constant in the denominator.

Finally, we recall that this bound is for homogeneous quasi-arithmetic progressions with largest term at most $n(t+1)$. This implies that for largest term at most n, we can obtain discrepancy at least

$$\frac{(\log_2 \frac{n}{t+1})^{1/4}}{23.6\sqrt{t}}$$

which, for large enough n, gives

$$|D(x,y)| \geq \frac{(\log_2 n)^{1/4}}{24\sqrt{t}}.$$

This completes the proof of Theorem 2.1. ∎

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